Level correlations in integrable systems

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We derive a simple analytical expression for the level correlation function of an integrable system. It accounts for both the lack of correlations at smaller energy scales and for global rigidity (level number conservation) at larger scales. We apply our results to a rectangle with incommensurate sides and show that they are in excellent agreement with the limiting cases established in the semiclassical theory of level rigidity.

I. INTRODUCTION

In the quantum limit, classically integrable systems are often described as those with "level bunching" or, alternatively, absence of level correlations, while classically chaotic ergodic systems are described as exhibiting "level rigidity" [1]. We will argue here, however, that such descriptions are overly simplistic and apply only in certain energy intervals. In fact, since the total number of levels must be conserved, both integrable and chaotic systems exhibit various degrees of rigidity depending on the energy scales.

The main difference between the chaotic and integrable systems lies in the relevant length and energy scales. Chaotic systems are characterized by their diffusive behavior [2] and the relevant energy scales are the mean level spacing Δ, the Thouless energy $E_c = D/L^2$, and the inverse scattering time $\tau^{-1} = D/\ell^2$, where $D \sim v_F \ell$ is the diffusion coefficient, v_F is the Fermi velocity, ℓ is the mean-free-path, and L is the system size ($\hbar = c = 1$ below). The Thouless scale is associated with the diffusion to the system boundary and can be written as[1] $E_c \sim \sqrt{\epsilon_F \Delta} \, (\ell/L)$, where ϵ_F is the Fermi energy.

For chaotic systems, on energy scales $E \ll \Delta$, the level rigidity statistics Δ_3, which is based on the cumulative level distribution function (see e.g. [1]), is

$$\Delta_3 \sim \frac{E}{\Delta} \tag{1}$$

indicating the lack of level correlations. The onset of rigidity occurs at $E \sim \Delta$ (and, hence, also known as level repulsion) and for $\Delta \ll E \ll E_c$

$$\Delta_3 \sim \ln \frac{E}{\Delta} \tag{2}$$

This energy range corresponds to length scales between L and $\sqrt{D/\Delta}$ and the rigidity is related to the diffusive character of the multiple traversals of the system. On scales such that $E_c \ll E \ll \tau^{-1}$, which correspond to lengths between ℓ and L and to uncorrelated diffusion in different parts of the system, the level structure is much less rigid [3],

$$\Delta_3 \sim \frac{E}{E_c} \sim \frac{1}{g} \frac{E}{\Delta} \tag{3}$$

where $g \sim E_c/\Delta \gg 1$ is the dimensionless conductance. The existence of the above regimes is, nonetheless, in complete agreement with the global rigidity (level number conservation) which can be expressed in the following form:

$$\int_{-\infty}^{\infty} d\omega \, \langle \delta\nu(\epsilon) \, \delta\nu(\epsilon + \omega) \rangle = 0 \tag{4}$$

where $\nu(\epsilon)$ is the level density, $\delta\nu(\epsilon) = \nu(\epsilon) - \langle\nu\rangle$ and $\langle\nu\rangle$ the mean level density.(Here ϵ is the "running" value of energy which below will be used interchangeably with ϵ_F). This relationship has been discussed, for instance in [4],

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[1] In what follows, we consider only 2D systems.

but without taking account of diffusion modes. The diffusion modes are only known to be treated perturbatively, but even then the above relationship is satisfied as can be easily seen with the help of eq. (32) in [3].

In integrable systems (4) must also be satisfied; otherwise even the notion of the mean level density Δ would be dubious. However, the only relevant scale in this case is

$$\varepsilon \sim \sqrt{\epsilon_F \Delta} \tag{5}$$

In what follows, we shall derive the expression for the level correlation function $\langle \delta\nu(\epsilon) \, \delta\nu(\epsilon + \omega) \rangle$ in integrable systems that satisfies (4) and illustrate it on an example of a rectangle with incommensurate sides (a generic integrable system). Applying this expression to evaluation of the Δ_3-statistics, we will find that for $E \ll \varepsilon$

$$\Delta_3 \sim \frac{E}{\Delta} \left(1 - O\left(\left(\frac{E}{\varepsilon} \right)^3 \right) \right) \tag{6}$$

and for $E \gg \varepsilon$

$$\Delta_3 \sim \frac{\varepsilon}{\Delta} \left(1 - O\left(\frac{\varepsilon^2}{E^2} \right) \right) \tag{7}$$

that is E-independent. The latter indicates that the spectrum becomes more rigid at large scales, the fact obviously related to (4). The leading terms in (6) and (7) were a subject of extensive numerical and analytical study in [5] and [6]. However, the evaluation in the present paper of the level correlation function over the entire energy range allows for the evaluation over the entire energy range of Δ_3- and Σ-statistics (and other characterizations of the spectrum [1]) also.

II. LEVEL CORRELATION FUNCTION

We approach the derivation of the level-correlation function in two different ways, which will later be shown to be equivalent. First, we utilize the semiclassical formalism wherein the level density is expressed as a sum over periodic orbits [6], [1]. In such a formalism, the Fourier transform of the level correlation function is given by

$$\phi(t) = \sum_j A_j^2 \delta(t - T_j) \tag{8}$$

where A_j and T_j are the periodic orbit amplitudes and periods respectively [6]. To simplify the calculation, we will express all energies in terms of Δ so that Δ will be dropped below. It is known that (see (58) in [6])

$$\phi(t \to \infty) = 1/2\pi \tag{9}$$

and it is clear that $\phi(t) = 0$ for $t < T_{\min}$, where $T_{\min}$ ($\sim \varepsilon^{-1}$ in 2D) is the shortest periodic orbit. Based on these limiting behaviors, we propose a very simple ansatz, namely,

$$\phi(t) = \begin{array}{ll} 1/2\pi, & t > T_{\min} \\ 0, & t < T_{\min} \end{array} \tag{10}$$

Clearly, this ansatz is applicable in any dimension.

Since the correlation function should be symmetrical with respect to the $\omega \to -\omega$ transformation, the FT should be symmetrical with respect to the $t \to -t$ transformation. The obvious generalization of (8) would be via the substitution $\delta(t - T_j) \to [\delta(t - T_j) + \delta(t + T_j)]$ and the ansatz (10) is generalized be means of

$$\phi(t) = \frac{1}{2\pi} \left[1 + \frac{\text{sign}(t - T_{\min})}{2} - \frac{\text{sign}(t + T_{\min})}{2} \right] \tag{11}$$

which satisfies $\phi(-t) = \phi(t)$. With such definition, the correlation function becomes

$$\langle \delta\nu(\epsilon) \, \delta\nu(\epsilon + \omega) \rangle = \delta(\omega) - \frac{\sin(\omega T_{\min})}{\pi\omega} \tag{12}$$

Consider now a rectangle whose sides L_1 and L_2 are such that

2

$$L_1^2/L_2^2 = \alpha \tag{13}$$

is irrational. It is also assumed that $\alpha \lesssim 1$ (this assumption is opposite to the assumption $\alpha \gtrsim 1$ in [5], [6]). We also have

$$\epsilon_F = N\Delta = \frac{2\pi}{mA} \tag{14}$$

where $A = L_1 L_2$ is the area and N is the mean number of levels below the Fermi level. With these definitions, the shortest periodic orbit is the one with length $2L_1$ and

$$T_{\min} = \frac{2L_1}{v_F} = \frac{2\pi^{1/2}\alpha^{1/4}}{\sqrt{\epsilon_F}} \tag{15}$$

The level correlation function for this integrable system is then

$$\langle \delta\nu\left(\epsilon\right)\delta\nu\left(\epsilon+\omega\right)\rangle = \delta\left(\omega\right) - \frac{\sin\left(2\pi\omega/\varepsilon\right)}{\pi\omega} \tag{16}$$

with the definition

$$\varepsilon = \sqrt{\pi\epsilon_F}/\alpha^{1/4} \tag{17}$$

(compare this definition with eqs. (7) and (43) in [6]).

The alternative way of derivation is specific to a rectangle (since it is based on the energy spectrum specific to the rectangular box) and involves a slight modification of a derivation for a square in Appendix B of [7]. It is shown there that the level density can be written as

$$\delta\nu\left(\epsilon\right) = \sqrt{\frac{2}{\pi}} \sum_{m_1,m_2=-\infty}^{\infty} \frac{\cos\left(kL_\sigma\right)}{\left(kL_\sigma\right)^{1/2}} \tag{18}$$

The term $m_1 = m_2 = 0$ is excluded. (In fact, this term gives the mean level density [7]). In (18), $k = \sqrt{2m\epsilon}$, $kL \gg 1$ and

$$L_\sigma = 2\sqrt{\left(m_1 L_1\right)^2 + \left(m_2 L_2\right)^2} \tag{19}$$

Notice that reducing the summation to only positive m_1, m_2 demonstrates that (18) is a summation over periodic orbits whose length is given by (19). Omitting the rapidly oscillating terms, we find

$$\langle \delta\nu\left(\epsilon\right)\delta\nu\left(\epsilon+\omega\right)\rangle = \frac{1}{\pi}\sum \frac{\cos\left(\Delta kL_\sigma\right)}{\left(kL_\sigma\right)^{1/2}} \tag{20}$$

The sum can be converted to an integral in polar coordinates,

$$\langle \delta\nu\left(\epsilon\right)\delta\nu\left(\epsilon+\omega\right)\rangle = \frac{1}{2kL_1 L_2}\left[\int_{-\infty}^{\infty}\exp\left(2i\Delta k\rho\right)d\rho - \int_{-\rho_{\min}}^{\rho_{\min}}\cos(2\Delta k\rho)d\rho\right] \tag{21}$$

where[2] $\rho_{\min} \approx L_1$. The appearance of $\rho_{\min}$ above is related to the fact that the term $m_1 = m_2 = 0$ is excluded in (18). It has the meaning of the length of the shortest periodic orbit, which is in direct relation to ansatz (11). (Notice that when $\alpha \sim 1$, $\rho_{\min} \approx L_1 \approx L_2$ and the combination of the cut-off and polar coordinates becomes a more tenable approximation). Integrating (21) and substituting $\Delta k = \omega m/k$, we recover the previously obtained result (16).

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[2]More consistently, the variable of integration should be a dimensionless $L_1\kappa \ll 1$; introducing the latter, however, does not affect the final result.

III. Δ_3- AND Σ-STATISTICS

We now proceed to apply (16) to the evaluation of Δ_3- and Σ-statistics. For Δ_3-statistics, we begin with

$$\Delta_3(E) = 2 \int_0^\infty \frac{dt}{t^2} \phi(t) G(Et/2) \tag{22}$$

$$G(y) = 1 - F^2(y) - 3(F'(y))^2, \ F(y) = \sin y / y$$

(compare with eq. (20) in [6]). Substituting (11) into (22), we find

$$\Delta_3(E) = \frac{E}{2\pi} \int_{ET_{\min}/2}^\infty dt \frac{G(t)}{t^2} \tag{23}$$

This is easily evaluated analytically in terms of algebraic, trigonometric and sine integral functions. However, of greater interest are the limiting cases:

$$\Delta_3 \left(E \ll T_{\min}^{-1} \sim \varepsilon \right) \cong \frac{1}{15} E \left[1 - \frac{(T_{\min}E)^3}{144\pi} \right] \tag{24}$$

and

$$\Delta_3 \left(E \gg T_{\min}^{-1} \sim \varepsilon \right) \cong \frac{1}{\pi T_{\min}} \left[1 - \frac{8}{3(T_{\min}E)^2} \right] \tag{25}$$

For the rectangular, upon substitution of (15), the first term in (25) becomes

$$\Delta_3^{(rect)} (E \gg \varepsilon) = \frac{1}{2\pi^{3/2} \alpha^{1/4}} \sqrt{\epsilon_F} \tag{26}$$

Remarkably, for $\alpha \sim 1$, the constant in front of the radical is close (~ 5 percent) to the value in (45) of [6] which was obtained by exact summation over all periodic orbits. This supports the utility of the ansatz (11) for the level correlation function.

For completeness we give the results for Σ-statistics as well. In the same limiting cases, we find

$$\Sigma \left(E \ll T_{\min}^{-1} \sim \varepsilon \right) \cong E \left[1 - \frac{(T_{\min}E)}{\pi} \right] \tag{27}$$

and

$$\Sigma \left(E \gg T_{\min}^{-1} \sim \varepsilon \right) \cong \frac{2}{\pi T_{\min}} \left[1 + \frac{\sin(T_{\min}E)}{T_{\min}E} - \frac{2\cos(T_{\min}E)}{(T_{\min}E)^2} \right] \tag{28}$$

We observe that the accuracy of Δ_3 in the limiting cases is higher than that of Σ. This is because Δ_3 describes the cumulative behavior of the levels in the spectrum and is thus a more appropriate characteristic for the rigidity. We also observe that the last two terms in (28) are rapidly oscillating. The most interesting feature of both statistics for large energy intervals is that the leading terms are E-independent (but do depend on ϵ).

IV. DISCUSSION

The central result of this work is (12). This simple form of the correlation function is in excellent agreement with the limiting behavior of the Δ_3-statistics for both small and large energy scales and is consistent with the overall spectral rigidity (4). Having a closed-form correlation function, applicable at all energy scales, allows for the evaluation of the closed-form expressions for Δ_3- and Σ-statistics as well, with the corrections in the appropriate regimes given by eqs. (24)-(25) and eqs. (27)-(28) respectively.

The significance of the corrections to the leading terms in (24) and (27) is that they describe the onset of level correlations at small scales, which are dominated by the δ-function in (12) (corresponding to the absence of correlations in the zeroth order). The corrections to the leading terms at large scales, (25) and (28), are also of interest especially

in view of the issues related to the symmetry-breaking perturbations (see below). For verification purposes, we are presently working on the numerical evaluation of Σ-statistics which will be reported elsewhere.

We emphasize that (12) was obtained both via ansatz (11) and, for the specific case of a rectangle with incommensurate sides, by a direct evaluation in the energy space. In both circumstances the small inaccuracy is related to the continuous nature of the approximation used versus the discrete behavior in the limit of short orbits.

Clearly, if a term $\delta\phi\left(t\right)$, such that $\delta\phi\left(T < T'_{\min}\right) = 0$ and $\delta\phi\left(\pm\infty\right) = 0$, is added to (11), eq. (4) will still be satisfied and the leading terms in (24) and (27), determined by the δ-function in (12), will remain unchanged. Furthermore, since the relevant scales are $T'_{\min} \gtrsim T_{\min}$, the corrections introduced at large energy scales will have the same functional form as, and will be of the order of, (25) and (28). The question of $\delta\phi\left(t\right)$ may be relevant in discussion of the effect of time-reversal symmetry-breaking terms, such as magnetic field. We hope to address this question in a future publication.

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